Generalized Laws of Black Hole Thermodynamics and Quantum Conservation Laws on Hawking Radiation Process

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Four classical laws of black hole thermodynamics are extended from exterior (event) horizon to interior (Cauchy) horizon. Especially, the first law of classical thermodynamics for Kerr-Newman black hole (KNBH) is generalized to those in quantum form. Then five quantum conservation laws on the KNBH evaporation effect are derived in virtue of thermodynamical equilibrium conditions. As a by-product, Bekenstein-Hawking's relation $S = A/4$ is exactly recovered.

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Due to the celebrated works of Hawking [1] and Bekenstein [2], black holes are demonstrated to be thermodynamic objects endowed with a temperature and an entropy. This put the first laws of black hole thermodynamics on a solid fundament [3]. Despite considerable effort [4] about the quantum [5], dynamic [6], or statistical [7]

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origin of black hole thermodynamics, however, the exact source and mechanism of the Benkenstein-Hawking entropy remain unclear [8].

In this paper, we assume that the whole space-time of a rotating charged black hole is described by Kerr-Newman metric and base our discussion on a sourceless charged massive scalar field on Kerr-Newman black hole (KNBH) background in the non-extreme case ($0 < \varepsilon = \sqrt{M^2 - a^2 - e^2} \leq M$). We propose that there should exist an inner thermal radiation on the internal horizon and that a Kerr-Newman black hole have a pair of entropy accompanied by a pair of temperature. Under this assumption, we can extend Bardeen-Carter-Hawking's four laws of black hole thermodynamics on exterior horizon to those on internal horizon. We also suggest a pair of quantum first laws of thermodynamics. Then quantum conservation laws on black hole radiation process are derived from the point of view of thermodynamical equilibrium [9]. Finally, the relation between classical entropy and quantum entropy of black hole is briefly discussed.

1. Preliminary: Separation of Covariant Klein-Gordon Equation on KNBH Background

In the Boyer-Lindquist coordinates, Kerr-Newman metric is given by [10,11]:

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}[adt - (r^2 + a^2)d\varphi]^2 + \Sigma(\frac{dr^2}{\Delta} + d\theta^2) \qquad (1)$$

with $\Delta = r^2 - 2Mr + a^2 + e^2 = (r - r_+)(r - r_-), \Sigma = r^2 + a^2\cos^2\theta, r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$, where mass M, charge e and specific angular momentum $a = J/M$ being three parameters of KNBH. (In Planck unit system $G = \hbar = c = k_B = 1$).

In the KNBH geometry, a complex scalar field Φ with mass μ and charge q satisfies the following covariant Klein-Gordon equation (KGE) [10,12]:

$$\begin{aligned} &\frac{-1}{\Delta}[(r^2 + a^2)\partial_t + a\partial_\varphi + iqer]^2\Phi + \partial_r(\Delta\partial_r\Phi) - \mu^2\Sigma\Phi \\ &+(a\sin\theta\partial_t + \frac{1}{\sin\theta}\partial_\varphi)^2\Phi + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) = 0. \end{aligned} \qquad (2)$$

The wave function Φ of the above equation has a solution of variables separable form $\Phi(t,r,\theta,\varphi) = R(r)S(\theta)e^{i(m\varphi-\omega t)}$ [11], in which the separated radial and angular part of KGE can be given as follows [12]:

$$\frac{1}{\sin\theta}\partial_\theta[\sin\theta\partial_\theta S(\theta)] + [\lambda - \frac{m^2}{\sin^2\theta} + (\mu^2-\omega^2)a^2\sin^2\theta]S(\theta) = 0, \tag{3}$$

$$\partial_r[\Delta\partial_r R(r)] + [\frac{K^2(r)}{\Delta} - \mu^2(r^2+a^2) - \lambda + 2ma\omega]R(r) = 0, \tag{4}$$

here λ is a separation constant, and $K(r) = \omega(r^2+a^2) - qer - ma$.

The general solutions to the angular equation of Eq.(3) are ordinary spheroidal angular wave functions [13] with spin-weight $s = 0$, while the radial equation of Eq.(4) can be reduced to the following generalized spin-weighted spheroidal wave equation [14] of imaginary number order (see Eqs.(13) and (14) in Ref. [12]):

$$\begin{aligned}&\partial_r[(r-r_+)(r-r_-)\partial_r R(r)] + [k^2(r-r_+)(r-r_-) + 2D(r-M) + \\ &\frac{[A(r-M)+\varepsilon B]^2}{(r-r_+)(r-r_-)} + (2\omega^2-\mu^2)(2M^2-e^2) - 2qeM\omega - \lambda]R(r) = 0,\end{aligned} \tag{5}$$

where we have put

$$A = 2M\omega - qe, \varepsilon B = \omega(2M^2-e^2) - qeM - ma, D = A\omega - M\mu^2, k^2 = \omega^2 - \mu^2.$$

Introducing $w_\pm = (B \pm A)/2$ for simplicity, and making substitution $R(r) = (r-r_+)^{i(B+A)/2}(r-r_-)^{i(B-A)/2}F(r)$, we can transform Eq.(5) for $R(r)$ into a modified generalized spheroidal wave equation with an imaginary spin-weight iA for $F(r)$ [12]:

$$\begin{aligned}&(r-r_+)(r-r_-)\partial_r^2 F(r) + 2[i\varepsilon A + (1+iB)(r-M)]\partial_r F(r) + [k^2(r-r_+)(r-r_-) \\ &+2D(r-M) + (2\omega^2-\mu^2)(2M^2-e^2) - 2qeM\omega - \lambda + A^2 - B^2 + iB]F(r) = 0.\end{aligned} \tag{6}$$

Eq.(6) has two regular singular points $r = r_\pm$ whose indices are $\rho_+ = 0, -2iw_+$ and $\rho_- = 0, -2iw_-$ respectively. The general solutions to Eq.(6) have forms around regular

singular points $r_\pm$

$$F_+(r) = c_1 f_1(A,B,D,k,r-r_+) + c_2(r-r_+)^{-2iw_+} g_1(A,B,D,k,r-r_+), \quad (7)$$

$$F_-(r) = d_1 f_2(A,B,D,k,r-r_-) + d_2(r-r_-)^{-2iw_-} g_2(A,B,D,k,r-r_-) \quad (8)$$

where functions f_1, f_2 are first solutions to Eq.(6), while g_1, g_2 being linear independent second ones to it. They are four sets of orthonormal generalized spheroidal wave functions being regular at points r_+, r_- respectively. We choose such an eigenvalue λ that makes $F_\pm(r)$ finite at $r = r_\pm$ respectively, namely functions f_1, f_2 and g_1, g_2 are regular over their corresponding regions.

The physical domain for radial coordinate r is $[0,\infty) = [0,r_-)\cup(r_-,r_+)\cup(r_+,\infty)$, with region $[0,\mu^2/2)\cup[\mu^2/2,\mu^2]\cup(\mu^2,\infty)$ for quadratic energy ω^2. We can extend simultaneously both intervals on real axes for coordinate and that for energy to corresponding whole complex planes including real axes $(-\infty,+\infty)$. The extreme case $\varepsilon = 0$ and special cases $\omega = \pm\mu/\sqrt{2}$, as well as $\omega = \pm\mu$ need to be carefully dealt with, but we don't discuss it here.

2. Hawking Radiation: External or Internal?

The exterior horizon and interior horizon, denoted by $\mathcal{H}_\pm$, are located at points $r_\pm = M \pm \varepsilon$. We shall consider a wave outgoing from horizon $\mathcal{H}_+$ over intervals $r_- < r < r_+$ and $r_+ < r < \infty$.

According to the method of Damour and Ruffini's [15], a correct outgoing wave $\Phi^{\rm out} = \Phi^{\rm out}(t,r,\theta,\varphi)$ is an adequate superposition of functions $\Phi^{\rm out}_{r>r_+}$ and $\Phi^{\rm out}_{r<r_+}$:

$$\Phi^{\rm out} = C_+[\eta(r-r_+)\Phi^{\rm out}_{r>r_+} + \eta(r_+-r)\Phi^{\rm out}_{r<r_+} e^{2\pi w_+}] \quad (9)$$

where $\eta(x)$ is conventional unit step function, with the outgoing wave components $\Phi^{\rm out}_{r>r_+}$ and $\Phi^{\rm out}_{r<r_+}$ being given by

$$\Phi^{\rm out}_{r>r_+}(t,r,\theta,\varphi) \sim c_1(r-r_+)^{iw_+}(r-r_-)^{iw_-} f^\ell_{w_+,w_-}(k,r) S^\ell_{m,0}(ka,\theta) e^{i(m\varphi-\omega t)}, \quad (10)$$

$$\Phi^{\rm out}_{r<r_+}(t,r,\theta,\varphi) \sim c_2(r-r_+)^{-iw_+}(r-r_-)^{iw_-}g^{\ell}_{w_+,w_-}(k,r)S^{\ell}_{m,0}(ka,\theta)e^{i(m\varphi-\omega t)}, \qquad (11)$$

here regular function $f^{\ell}_{w_+,w_-}(k,r) = f_1$ is the first solution to the generalized spheroidal radial wave equation [12,14], and regular function $g^{\ell}_{w_+,w_-}(k,r) = g_1$ is the second one to the same equation, while $S^{\ell}_{m,0}(ka,\theta)$ is an ordinary spheroidal angular wave function [13]. These functions can be orthonormalized to constitute their corresponding orthogonal complete functions [12,13,14]. In addition, scalar wave function Φ has asymptotic behaviors of plane waves at infinity:

$$\Phi(t,r,\theta,\varphi) \rightarrow e^{i(\pm kr-\omega t+m\varphi)}S^{\ell}_{m,0}(ka,\theta), \qquad (r \rightarrow \pm\infty)$$

In fact, components $\Phi^{\rm out}_{r>r_+}$ and $\Phi^{\rm out}_{r<r_+}$ have asymptotic behaviors when $r \rightarrow r_+$

$$\Phi^{\rm out}_{r>r_+} \rightarrow c_1(r-r_+)^{iw_+}S^{\ell}_{m,0}(ka,\theta)e^{i(m\varphi-\omega t)}, \quad (r \rightarrow r_+) \qquad (12)$$

$$\Phi^{\rm out}_{r<r_+} \rightarrow c_2(r-r_+)^{-iw_+}S^{\ell}_{m,0}(ka,\theta)e^{i(m\varphi-\omega t)}, \quad (r \rightarrow r_+) \qquad (13)$$

Clearly, the outgoing wave $\Phi^{\rm out}_{r>r_+}$ can't be directly extended from $r_+ < r < \infty$ to $r_- < r < r_+$, but it can be analytically continued to an outgoing wave $\Phi^{\rm out}_{r<r_+}$ that inside event horizon $\mathcal{H}_+$ by the lower half complex r-plane around unit circle $r = r_+ - i0$:

$$r - r_+ \rightarrow (r_+ - r)e^{-i\pi}.$$

By this analytical treatment, we have

$$\Phi^{\rm out}_{r<r_+}(t,r,\theta,\varphi) \sim c_2(r-r_+)^{-iw_+}(r-r_-)^{iw_-}f^{\ell}_{w_+,w_-}(k,r)S^{\ell}_{m,0}(ka,\theta)e^{i(m\varphi-\omega t)} \qquad (14)$$

here function $f^{\ell}_{w_+,w_-}(k,r)$ can be analytically continued to be function $g^{\ell}_{w_+,w_-}(k,r)$.

As a difference factor $(r-r_+)^{-2iw_+}$ emerges between functions $F^{\ell}_{w_+,w_-}(k,r) = (r-r_+)^{iw_+}(r-r_-)^{iw_-}f_1$ and $G^{\ell}_{w_+,w_-}(k,r) = (r-r_+)^{-iw_+}(r-r_-)^{iw_-}g_1$, then $\Phi^{\rm out}_{r>r_+}$ differs $\Phi^{\rm out}_{r<r_+}$ by a factor $e^{2\pi w_+}$, thus we can derive a relation

$$|\frac{\Phi^{\rm out}_{r>r_+}}{\Phi^{\rm out}_{r<r_+}}|^2 = e^{-4\pi w_+}. \qquad (15)$$

Using the method of Damour-Ruffini's, it is easy to obtain an "external" thermal radiation spectrum [15]:

$$\langle N_+ \rangle = |C_+|^2 = \frac{1}{e^{4\pi w_+} - 1}. \tag{16}$$

Similarly, due to symmetry between exterior horizon and interior horizon, we can also establish an "internal" black body spectrum:

$$\langle N_- \rangle = |C_-|^2 = \frac{1}{e^{4\pi w_-} - 1} \tag{17}$$

for a "right" outgoing wave $\Psi^{\text{out}} = \Psi^{\text{out}}(t, r, \theta, \varphi)$ from horizon $\mathcal{H}_-$ which is similar to the above-head "left" outgoing wave Φ^{out},

$$\Psi^{\text{out}} = C_-[\eta(r_- - r)\Psi^{\text{out}}_{r<r_-} + \eta(r - r_-)\Psi^{\text{out}}_{r>r_-} e^{2\pi w_-}] \tag{18}$$

and

$$|\frac{\Psi^{\text{out}}_{r<r_-}}{\Psi^{\text{out}}_{r>r_-}}|^2 = e^{-4\pi w_-}. \tag{19}$$

Now, we have made analytical extension for Ψ^{out} by the upper half complex r-plane around unit circle $r = r_- + i0$:

$$r_- - r \longrightarrow (r - r_-)e^{-i\pi}.$$

3. Generalized First Laws of Black Hole Thermodynamics: Classical and Quantum

It is convenient to introduce formally the following notations:

$$\begin{array}{ll}
\text{Reduced horizon area} : \mathcal{A}_\pm = r_\pm^2 + a^2, & \text{Horizon} : r_\pm = M \pm \varepsilon, \\
\text{Surface gravity} : \kappa_\pm = \frac{r_\pm - M}{\mathcal{A}_\pm} = \frac{\pm\varepsilon}{\mathcal{A}_\pm}, & \text{Angular velocity} : \Omega_\pm = \frac{a}{\mathcal{A}_\pm}, \\
\text{Electric potential} : \Phi_\pm = \frac{e r_\pm}{\mathcal{A}_\pm}, & \text{Frequency} : \omega_\pm = m\Omega_\pm + q\Phi_\pm.
\end{array}$$

We can derive algebraically the following generalized first laws of classical and quantum black hole thermodynamics in both integral and differential forms on exterior horizon $\mathcal{H}_+$ as well as on interior horizon $\mathcal{H}_-$ (see Appendix).

a. Generalized first laws of classical thermodynamics in differential and integral forms [3,16]:

$$dM = \frac{\kappa_\pm}{2} d\mathcal{A}_\pm + \Omega_\pm dJ + \Phi_\pm de, \tag{20}$$

$$M = \kappa_\pm \mathcal{A}_\pm + 2J\Omega_\pm + e\Phi_\pm. \tag{21}$$

b. Generalized first laws of quantum thermodynamics in integral and differential forms [9,17]:

$$\omega = 2\kappa_\pm w_\pm + m\Omega_\pm + q\Phi_\pm, \tag{22}$$

$$d\omega = 2\kappa_\pm dw_\pm + \Omega_\pm dm + \Phi_\pm dq. \tag{23}$$

Relations (20, 21) demonstrate that electro-magnetic energy $e\Phi_\pm$ is an interaction energy (gauge term), while term $\kappa_\pm \mathcal{A}_\pm$ and term $J\Omega_\pm$ being self energy terms.

4. Quantum Conservation Laws

Let us consider a complex scalar field Φ in thermal equilibrium with a Kerr-Newman black hole at a pair of local temperature $\mathcal{T}_\pm = \kappa_\pm/2$. In thermal equilibrium radiation process, surface gravity , angular velocity and electrical potential can be considered to undertake little change. In virtue of conditions that thermodynamical equilibrium could exist on horizons:

$$\kappa_{\pm-0} = \kappa_{\pm+0}, \Omega_{\pm-0} = \Omega_{\pm+0}, \Phi_{\pm-0} = \Phi_{\pm+0},$$

combining differential relations Eq.(20) with Eq.(23), we can deduce five quantum conservation laws for energy, angular momentum, charge and entropy respectively:

$$dM = n d\omega, \text{(Energy)} \tag{24}$$

$$dJ = n dm, \text{(Angular Momentum)} \tag{25}$$

$$de = n dq, \text{(Charge)} \tag{26}$$

$$\frac{1}{4} d\mathcal{A}_\pm = n dw_\pm, \text{(Entropy)}. \tag{27}$$

Here n is an integral multiplier. From integral relations (21) and (22), we can also obtain a special quantum state $nm = J, n\omega = M/2, nq = e/2, nw_\pm = \mathcal{A}_\pm/4$.

Eqs.(24-27) indicate that a Kerr-Newman black hole has discrete increment of energy, angular momentum, charge and entropy. When a KNBH is in dynamical equilibrium with a scalar field, it radiates the same quantity of quanta as that it absorbs. Thus the total quantities of energy, charge, entropy and angular momentum of the whole system being consisted of black hole and scalar field quanta remain unchanged in this thermodynamical equilibrium radiation process [9].

5. Entropy: Classical and Quantum

In fact, Eq.(27) is a pair of generalized second thermodynamic laws in quantum form. By integrating this equation, we obtain quantum black hole entropy:

$$nw_\pm = \frac{1}{4}\mathcal{A}_\pm + C_\pm. \tag{28}$$

As Bekenstein-Hawking's classical entropy [1,2] is $S_\pm = A_\pm/4 = \pi\mathcal{A}_\pm$, quantum entropy $nw_\pm$ are equivalent to the reduced entropy $\mathcal{S}_\pm = S_\pm/(4\pi)$, so we have Bekenstein-Hawking relations (Choose constant $C_\pm = 0$):

$$\mathcal{S}_\pm = nw_\pm = \mathcal{A}_\pm/4. \tag{29}$$

Eqs.(28, 29) demonstrate that Bekenstein-Hawking classical entropy origins statistically from quantum entropy of quantized field, that is, the classical entropy of black hole is equal to quantum entropy of field [9].

6. Spectrum: Continued or Discrete?

Quantum numbers of entropy $nw_\pm$ must be integers as quantities A, B, ε correspond to angular momentum $-m, -s, -a$ respectively when mass $\mu = 0$. It is suggested that the quantum entropy $nw_\pm$ be discrete numbers, namely be integers. The thermal spectrum $\langle N_\pm \rangle$ for bound states are discrete spectrum, while the spectrum for scattering states being continual ones.

7. Temperature: Positive or Negative?

From thermal spectrum of Hawking radiation:

$$\langle N_\pm \rangle = \frac{1}{e^{4\pi w_\pm} - 1},$$

$$w_\pm = \frac{\omega - m\Omega_\pm - q\Phi_\pm}{2\kappa_\pm}$$

we can deduce that a KNBH has a pair of local temperature $T_\pm = \kappa_\pm/(2\pi) = \mathcal{T}_\pm/\pi$ on horizons $\mathcal{H}_\pm$.

If we accept the temperature interpretation of surface gravity $\kappa_\pm = \pm\varepsilon/(r_\pm^2 + a^2)$, then temperature T_+ is positive while T_- being negative. The definition of negative temperature has no contradict with black hole having negative specific heat.

8. Generalized Four Thermodynamical Laws

We give the chief points of four generalized laws of black hole thermodynamics as follows:

The Zeroth Law: The surface gravity $\kappa_\pm$ of a stationary black hole (at equilibrium) are two constants on the entire surface of its corresponding horizons $\mathcal{H}_\pm$.

The First Law: In an isolated system including black holes, the total energy of the system is conserved.

The Second Law: The total entropy $S_{\mathrm{T}} = S_{\mathrm{BH}} + S_{\mathrm{M}}$, never decreases in any physical process, where S_{M} is the total entropy of ordinary matter outside black holes, $\delta S_{\mathrm{T}} \geq 0$.

The Third Law: It is impossible by any physical process to reduce $\kappa_\pm$ to zero by a finite sequence of operations. However, this can be violated by quantum vacuum fluctuations. Quantum evaporation effect can make a KNBH undertake a second order phase transition [18] from the non-extreme case $(M^2 \neq a^2 + e^2)$ to the extreme case $(M^2 = a^2 + e^2)$.

Quantum Conservation Laws: In a thermal equilibrium process of black hole radiation, the total energy, total charge, total angular momentum and total entropy of the

whole system are conserved.

To summarize, many results on the exterior horizon are generalized to similar ones on the inner horizon. We suggest that there should exist an interior radiation on the Cauchy horizon provided that the whole spacetime is described by the Kerr-Newman line element. This provides a rather good interpretation of the origin of black hole classical entropy arising statistically from quantum entropy of field quanta. Thus, if a KNBH really has two pairs of temperature and entropy, then how do we interpret them ? There exists a proposal that a KNBH be a two-energy levels system endowed with a pair of local temperature on the horizons. As far as existed theories are concerned, it seems to have no reason to exclude a negative temperature T_-.

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APPENDIX: Algebraical Derivation of First Laws of Thermodynamics in Classical and Quantum Forms

Derivation: By differentiating equality $\mathcal{A}_\pm = r_\pm^2 + a^2 = 2Mr_\pm - e^2$, we can deduce a relation $r_\pm dM = \pm\varepsilon dr_\pm + ada + ede$. Then multiplying this formula by $r_\pm$ and adding a term a^2dM, we obtain a relation $\mathcal{A}_\pm dM = \pm\varepsilon/2d\mathcal{A}_\pm + adJ + er_\pm de$. Eq.(20) is obtained by dividing this relation with $\mathcal{A}_\pm$.

From $r_\mp(\mathcal{A}_\pm + e^2) = 2Mr_+r_- = 2M(a^2 + e^2) = 2Ja + 2Me^2$, we can deduce a relation $(M \mp \varepsilon)\mathcal{A}_\pm = 2Ja + (2M - r_\mp)e^2 = 2Ja + e^2r_\pm$. Then Eq.(21) is obtained by dividing this equality with $\mathcal{A}_\pm$ and replacing terms $\kappa_\pm\mathcal{A}_\pm = \pm\varepsilon$.

From equalities

$$A = (\omega - \omega_+)/(2\kappa_+) - (\omega - \omega_-)/(2\kappa_-), B = (\omega - \omega_+)/(2\kappa_+) + (\omega - \omega_-)/(2\kappa_-)$$

we can obtain relations $w_\pm = (B \pm A)/2 = (\omega - \omega_\pm)/(2\kappa_\pm)$. Eq.(22) is obtained by multiplicating this relations with $2\kappa_\pm$ and displacing terms $\omega_\pm = m\Omega_\pm + q\Phi_\pm$. Then

by differentiating this equation, we obtain Eq.(23).

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